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On an Infinite Series of Abel Occurring in the Theory of Interpolation

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The purpose of this paper is to show that for a certain class of functions f which are analytic in the complex plane possibly minus $(-\infty, -1]$, the Abel series $f(0) + \sum_{n=1}^{\infty} f^{(n)}(n\beta)\, z(z - n\beta)^{n-1}/n!$ is convergent for all $\beta > 0$. Its sum is an entire function of exponential type and can be evaluated in terms of f. Furthermore, it is shown that the Abel series of f for small $\beta > 0$ approximates f uniformly in half-planes of the form $\mathrm{Re}(z) \geqslant -1 + \delta$, $\delta > 0$. At the end of the paper some special cases are discussed.

1. INTRODUCTION

Among the manuscripts of N. H. Abel which appeared for the first time in his Collected Work there is a paper with the title: "Sur les fonctions génératrices et leurs déterminantes" [1]. In this paper Abel discusses a number of expansion problems for a special class of functions. One of the expansion problems, which later turned out to belong to an important class of problems in the theory of interpolation of entire functions (see [3]), is the following: Given (complex) constants $\beta \neq 0$ and h, expand the function $f_h(x) = f(x + h)$ in an infinite series of polynomials in x with coefficients of the form $f^{(n)}(h + n\beta)$, $n = 0, 1, 2, \ldots$. In the attempt to solve this problem, Abel was led to expansions of the type

$$f(x + h) = f(h) + \sum_{n=1}^{\infty} (f^{(n)}(h + n\beta)/n!)\, x(x - n\beta)^{n-1} \tag{1.1}$$

which, for $h = 0$, reduce to the expansions

$$f(x) = f(0) + \sum_{n=1}^{\infty} (f^{(n)}(n\beta)/n!)\, x(x - n\beta)^{n-1}. \tag{1.2}$$

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For $\beta = 0$ the expansions (1.1) and (1.2) reduce to the classical ones of Taylor and McLaurin, respectively.

Abel did not discuss in any detail under what conditions on f the expansions (1.1) and (1.2) are valid except for the case that f is a polynomial.

The first detailed study of expansions of the form (1.1) and (1.2) was made by G. Halphen [6] and appeared in 1882. Halphen established criteria for convergence and observed that if $\beta \neq 0$ and the series (1.2) converges, then it represents a special entire function of exponential type. Furthermore, Halphen observed that, if f is a rational function, then the Abel series (1.2) converges for all complex x and for all values of β for which it is defined but that in that case the expansion never represents the function unless it is a polynomial.

It was not until 1935 that a complete solution of the convergence problem of series of the form (1.2) appeared in a paper by W. Gontcharoff [4] (see also [3] and [5]). In [4], W. Gontcharoff considered general infinite series, called *Abel series*, of the form

$$c_0 + \sum_{n=1}^{\infty} c_n z(z - n\beta)^{n-1}/n!, \tag{1.3}$$

where z is complex and the constants c_n ($c = 0, 1, 2,...$) may depend on β. If the coefficients c_n ($n = 0, 1, 2,...$) are of the form $f^{(n)}(n\beta)$ for some function f, then (1.3) is called the *Abel series generated by f*. The important results of Gontcharoff on Abel series contained in [4] can be summarized as follows.

THEOREM I. (*W. Gontcharoff*). *If $\beta \neq 0$ and if* (1.3) *converges for one value of $z \neq 0$, then it converges for all complex z and the convergence is uniform on compact subsets of the complex plane. Furthermore, if for $\beta \neq 0$ the infinite series converges absolutely for one value of $z \neq 0$, then it is absolutely convergent for all z.*

Concerning the properties of the class of functions determined by convergent Abel series Gontcharoff proved the following result.

THEOREM II. (*W. Gontcharoff*). *If f is an entire function of exponential type and if β is a complex constant such that the conjugate indicator diagram of f is contained in the interior of the compact convex domain G_β bounded by the set of points w satisfying $|(\beta w)\exp(1 + \beta w)| \leq 1$ and $\mathrm{Re}(\beta w) \geq -1$, then the Abel series generated by f converges to f. Conversely, if* (1.3) *is convergent for some $\beta \neq 0$, then its sum is an entire function of exponential type whose conjugate indicator diagram is contained in the domain G_β.*

For generalizations of Theorem II we refer the reader to [3, Chapter III].

As we indicated above Halphen observed in [6] that the Abel series generated by a rational function is convergent. Halphen did not discuss, however, the relationship, if any, between a rational function and the Abel series it generates. The main purpose of the present paper is an attempt to fill this gap. More precisely, we shall show that there exists a class of functions, which are analytic in the complex plane minus possibly the real numbers $\leqslant -1$, whose Abel series are convergent for all real β satisfying $\beta > 0$ and which have the property that for sufficiently small $\beta > 0$ they approximate the functions uniformly in half-planes of the form $\mathrm{Re}(z) \geqslant -1 + \delta$ $(\delta > 0)$. Examples are given to illustrate the theorem.

Before we start with a discussion of our main result we shall first present in the next section a simple direct proof of the first part of Gontcharoff's Theorem II. In doing so, we hope to be able to justify, in part, the method employed to obtain our main result.

2. Abel Series Generated by Entire Functions

We begin with the following preliminary observations.

It follows immediately from Rouché's theorem that if $\beta \neq 0$, then for each complex number t satisfying $e \mid \beta t \mid < 1$ the equation $z \exp(\beta z) = t$ has one and only one root $z = z(t)$ in the open disk $\{z\colon \mid z \mid < \mid 1/\beta \mid\}$. Then for each complex number a the function $\exp(az(t))$, $e \mid t\beta \mid < 1$ can be expressed as follows: $\exp(az(t)) = (1/2\pi i) \int \exp(a\xi)(F'(\xi)/F(\xi))\, d\xi$, where $F(\xi) = \xi \exp(\beta\xi) - t$ and the integration is over the circle $\mid \xi \mid = (1/\mid \beta \mid) - \epsilon$ for sufficiently small ϵ. Observing that

$$\exp(az(t)) = 1 + (1/2\pi i) \int \exp(a\xi)(\log(F(\xi)/(\xi \exp(\beta\xi)))'\, d\xi,$$

and by integrating the last integral by parts we obtain the well-known expansion

$$\exp(az(t)) = 1 + \sum_{n=1}^{\infty} a(a - n\beta)^{n-1}\, t^n/n!. \tag{2.1}$$

The expansion (2.1) is, in fact, a convergent Abel series in a for all t satisfying $e \mid \beta t \mid < 1$.

We replace now in (2.1) a by z and t by $w \exp(\beta w)$. Then we obtain the well-known expansion

$$\exp(zw) = 1 + \sum_{n=1}^{\infty} z(z - n\beta)^{n-1}\, w^n \exp(n\beta w)/n!. \tag{2.2}$$

The expansion (2.2) is a convergent Abel series in z for all $|\beta w| \leqslant 1$ satisfying $|\beta w \exp(1+\beta w)| \leqslant 1$.

For each $\beta \neq 0$ we shall denote by G_β the compact convex domain bounded by the set of points w satisfying $|\beta w \exp(1+\beta w)| = 1$ and $\operatorname{Re}(w\beta) \geqslant -1$. The function $z \exp(\beta z)$ maps the interior of G_β in a one-to-one fashion on the open disk around the origin of radius $1/|\beta e|$.

Assume now that f is an entire function of exponential type whose conjugate indicator diagram is contained in the interior of G_β and let $\varphi = \varphi(w)$ denote the analytic continuation of the Abel–Borel transform $(1/w)\int_0^\infty f(x/w)\exp(-x)\,dx$ of f. Then it is well-known that f can be expressed in terms of φ by means of the inversion formula

$$f(z) = (1/2\pi i)\int_\gamma \varphi(w)\, e^{zw}\, dw, \tag{2.3}$$

where γ is a simple closed contour containing the conjugate indicator diagram of f in its interior. Since, by hypothesis, the conjugate indicator diagram of f is contained in the interior of G_β we may assume that γ is contained in G_β. Hence, for such a contour γ, using the expansion (2.2), we obtain finally that uniformly on compact subsets of the complex plane the following expansion holds.

$$\begin{aligned} f(z) &= (1/2\pi i)\int_\gamma \varphi(w)\exp(zw)\,dw \\ &= \sum_{n=0}^{\infty} (z(z-n\beta)^{n-1}/n!)\cdot(1/2\pi i)\int_\gamma w^n \exp(n\beta w)\,\varphi(w)\,dw \\ &= \sum_{n=0}^{\infty} f^{(n)}(n\beta)\, z(z-n\beta)^{n-1}/n!, \end{aligned} \tag{2.4}$$

and the proof of the first part of Theorem II is completed.

Remark. In this context it is historically of interest to point out that Abel in his manuscript [1], referred to in the introduction, had already anticipated the importance of representing functions in the form (2.3). In fact, in [1] Abel considers functions f which can be written in the form $f(z) = \int \varphi(t)\exp(zt)\,dt$. For this reason we have called the function φ in (2.3) the Abel–Borel transform of f in place of the present terminology "the Borel transform of f."

By using expansion (2.2) Abel was able to derive (2.4) for functions f of the form $\int \varphi(t)\exp(zt)\,dt$. The derivation being completely formal did not, of course, enter into questions concerning its validity. Since Abel at that time also felt that every reasonable function could be written in the form $\int \varphi(t)\exp(zt)\,dt$ he applied (2.4) to functions other than entire functions such as $\log(1+z)$ for which the expansions (2.4) are not valid as was pointed out first by G. Halphen [6]. For a further discussion about this point we refer the reader to [6, p. 84] and to the Preface of [1].

In view of this it seems to be of interest to investigate the relationship between a function f, say $\log(1 + z)$, and the entire function which its Abel series generates. This will now be done in the next section.

3. Abel Series Generated by Certain Analytic Functions

We recall that G. Halphen in [6] observed that rational functions generate convergent Abel series. Of course, if the rational function is not a polynomial then its Abel series, representing an entire function, cannot coincide with it. We shall now show that there exists a class of functions more general than rational functions whose Abel series are convergent for some range of values of β. Furthermore, we shall be able to express the sum of the Abel series of f in terms of f. For the sake of simplicity we shall limit our discussion to a class of functions which are analytic in the complex plane possibly minus the set of real numbers $\leqslant -1$. More precisely, we define the following.

Definition (3.1). A function f analytic at $z = 0$ is said to be in the class $\mathfrak{A}$ if f can be written in the form

$$f(z) = z \int_0^1 1/(1 + zt)\, d\mu(t), \tag{3.2}$$

where μ is a (complex) Lebesgue–Stieltjes measure on [0, 1] of finite total variation.

It is obvious that if $f \in \mathfrak{A}$, then f is analytic for all complex values of z except possibly on the set of negative real numbers $\leqslant -1$.

Each $f \in \mathfrak{A}$ has a Taylor series expansion $\sum_{n=1}^{\infty} a_n z^n$, $|z| < 1$, with coefficients $a_n = \int_0^1 (-t)^{n-1}\, d\mu(t)$ $(n = 1, 2, \ldots)$.

The derivatives of f can be expressed in terms of μ by means of the following formulas

$$f^{(n)}(z) = n! \int_0^1 ((-t)^{n-1}/(1 + zt)^{n+1})\, d\mu(t). \tag{3.3}$$

It is easy to see that the functions $z/(a + z)$, $\mathrm{Re}(a) \geqslant 1$; $\log(1 + z)$, $(z)^{1/2} \arctan(z)^{1/2}$; $\sum_{n=1}^{\infty} (-1)^{n-1} z^n/n^2$, etc., are in $\mathfrak{A}$.

For every $f \in \mathfrak{A}$ we shall denote by $A_f(z; \beta)$ the formal Abel series generated by f, that is, in symbols

$$A_f(z; \beta) = \sum_{n=1}^{\infty} f^{(n)}(n\beta)\, z(z - n\beta)^{n-1}/n!. \tag{3.4}$$

Concerning the Abel series generated by functions in the class $\mathcal{A}$ we have the following theorem.

THEOREM (3.5). *For each $f \in \mathcal{A}$ and for each $\beta > 0$ the Abel series* (3.4) *generated by f is absolutely convergent and uniformly convergent on compact subsets of the plane.*

Proof. From (3.3) it follows that, if $\beta > 0$, then

$$| f^{(n)}(n\beta)/n! | = \left| \int_0^1 (-t)^{n-1}/(1 + nt\beta)^{n+1} \, d\mu(t) \right| \leqslant (2/e) \, \beta^{1-n} n^{-(n+1)} \cdot \int_0^1 d \, | \mu | \, .$$

Hence, for all $| z | \leqslant M$ and for all $n = 1, 2, \ldots$ we have

$$\begin{aligned} &| f^{(n)}(n\beta) \, z(z - n\beta)^{n-1}/n! \, | \\ &\leqslant (2/e) \, \beta^{1-n} n^{-(1+n)} \, M \cdot (n\beta)^{n-1} \, (1 + M/n\beta)^{n-1} \cdot \int_0^1 d \, | \mu | \\ &\leqslant (2M \cdot \exp((M/\beta - 1) \cdot \int_0^1 d \, | \mu \, |)/n^2, \end{aligned}$$

and the required result follows.

In the next lemma we shall present an expansion similar to (2.1) the proof of which is now left to the reader.

LEMMA (3.6). *Let $| ew | \leqslant 1$, $\beta > 0$, $0 < t \leqslant 1$ and let $\eta = \eta(w)$ be the unique root of the equation $\beta t \eta \exp(-\beta t \eta) = w$ satisfying $| \beta t \eta | \leqslant 1$. Then for all complex z*

$$\begin{aligned} &(\exp(-z\beta t\eta(w)) - 1)/t \\ &\qquad = - \sum_{n=1}^{\infty} (-t)^{n-1} \, \eta^n(w) \exp(-n\beta t \eta(w)) \cdot z(z - n\beta)^{n-1}/n! \, , \qquad (3.7) \end{aligned}$$

where the convergence of Abel's series on the right-hand side of (3.7) *is uniform in z on compact subsets of the complex plane.*

If in Lemma 3.6 we take w to be real, then it is easy to see that the equation $\xi \exp(-\xi) = w$ has exactly two real roots ξ_1, ξ_2 for all $0 \leqslant w \leqslant 1/e$ satisfying $0 \leqslant \xi_1 \leqslant 1 \leqslant \xi_2$. For $0 < \beta$ and $0 < t \leqslant 1$ we set $\xi_1 = \beta t \eta_1$ and $\xi_2 = \beta t \eta_2$. Then $0 \leqslant \eta_1 \leqslant 1/\beta t \leqslant \eta_2$ and $\beta t \eta_1 \exp(-\beta t \eta_1) = \beta t \eta_2 \exp(-\beta t \eta_2) = w$. Observe now that, if we substitute in the right-hand side of (3.7) for η the roots η_1 and η_2, respectively, then the result will be the same. Hence, we have the following result.

LEMMA (3.8). *For each* $0 < \beta$ *and for each* $0 < t \leqslant 1$, *if* η_1, η_2 *are the unique positive real numbers satisfying*

$$0 \leqslant \beta t \eta_1 \leqslant 1 \leqslant \beta t \eta_2 \qquad \text{and} \qquad \beta t \eta_1 \exp(-\beta t \eta_1) = \beta t \eta_2 \exp(-\beta t \eta_2),$$

then the following formulas hold uniformly in z on compact subsets of the complex plane.

$$(1/t)(\exp(-z\eta_1 t) - 1)\exp(-\eta_1) = -\sum_{n=1}^{\infty} (-1)^{n-1} z(z - n\beta)^{n-1} t^{n-1}\eta_1{}^n \exp(-(nt\beta + 1)\eta_1)/n!\,, \tag{3.9}$$

and

$$(1/t)(\exp(-z\eta_1 t) - 1)\exp(-\eta_2) = -\sum_{n=1}^{\infty} (-1)^{n-1} z(z - n\beta)^{n-1} t^{n-1}\eta_1{}^n \exp(-(nt\beta + 1)\eta_2)/n!\,. \tag{3.10}$$

For every $x \geqslant 1$ we shall denote by $\omega = \omega(x)$ the unique root of the equation $\xi \exp(-\xi) = x \exp(-x)$ satisfying $0 \leqslant \omega(x) \leqslant 1$. Then $\omega = \omega(x)$ is a decreasing function of x which decreases from one to zero as x increases from one to infinity. Now it is obvious that in Lemma 3.8 the root η_1 may be considered to be a function of η_2 and, in fact, η_1 can be expressed in terms of η_2 be means of the formula $\eta_1 = \omega(\beta t \eta_2)/\beta t$ ($\beta > 0$, $0 < t \leqslant 1$). With this additional notation we are now in a position to prove the main result of the paper.

THEOREM (3.11). *For each* $f \in \mathfrak{A}$ *and for each* $\beta > 0$ *we have for all z satisfying* $\mathrm{Re}(z) > -1$.

$$\begin{aligned} A_f(z;\beta) &= f(z) - \int_0^1 (1/t^2)\Big(\int_{1/\beta}^{\infty} (\exp(-z\omega(\beta x)/\beta) \\ &\qquad - \exp(-zx))\exp(-x/t)\,dx\Big)\,d\mu(t) \\ &= f(z) + \exp(-z/\beta)\int_0^1 (\exp(-1/\beta t))/t(1 + zt)\,d\mu(t) \\ &\qquad - \int_0^1 (1/t^2)\Big(\int_{1/\beta}^{\infty} \exp(-z\omega(\beta x)/\beta - x/t)\,dx\Big)\,d\mu(t). \end{aligned} \tag{3.12}$$

Furthermore, if $0 < \delta < 1$, *then for all z satisfying* $\mathrm{Re}(z) \geqslant -1 + \delta$ *we have the following estimate*

$$|\,A_f(z;\beta) - \theta(z)| \leqslant (1/\delta)\int_0^1 (\exp(-\delta/\beta t))/t)\,d\,|\,\mu\,|(t). \tag{3.13}$$

In particular, $A_f(z;\beta)$ tends to f uniformly in $\operatorname{Re}(z) \geqslant -1+\delta$, $\delta > 0$ *as β tends to zero.*

Proof. From (3.9) and (3.10) it follows easily that if $\operatorname{Re}(z) > -1$, then

$$\int_0^1 -\Big(\int_0^{1/\beta t} (1/t)(\exp(-zt\eta_1) - 1)\exp(-\eta_1)\,d\eta_1$$

$$+\int_{1/\beta t}^{\infty} (1/t)(\exp(-zt\eta_1) - 1)\exp(-\eta_2)\,d\eta_2\Big)\,d\mu(t)$$

$$= \int_0^1 \Big(\int_0^{\infty} \sum_{n=1}^{\infty} ((-1)^{n-1} z(z-n\beta)^{n-1} t^{n-1}\eta^n \exp(-(nt\beta+1)\eta))/n!\Big)\,d\eta\Big)\,d\mu(t)$$

$$= \int_0^1 \Big(\sum_{n=1}^{\infty} (-1)^{n-1} z(z-n\beta)^{n-1} t^{n-1}/(1+nt\beta)^{n+1}\Big)\,d\mu(t) = A_f(z;\beta);$$

and

$$f(z) = z\int_0^1 (1/(1+zt))\,d\mu(t)$$

$$= \int_0^1 -\Big(\int_0^{\infty} (1/t)(\exp(zt\eta) - 1)\exp(-\eta)\,d\eta\Big)\,d\mu(t)$$

$$= \int_0^1 -\Big(\int_0^{1/\beta t} (1/t)(\exp(-zt\eta_1) - 1)\exp(-\eta_1)\,d\eta_1$$

$$+\int_{1/\beta t}^{\infty} (1/t)(\exp(-zt\eta_2) - 1)\exp(-\eta_2)\,d\eta_2\Big)\,d\mu(t).$$

Hence,

$$A_f(z,\beta) = f(z)$$

$$-\int_0^1 \Big(\int_{1/\beta t}^{\infty} (1/t)(\exp(-zt\eta_1) - \exp(-zt\eta_2))\exp(-\eta_2)\,d\eta_2\Big)\,d\mu(t).$$

Replacing η_1 by $\eta_1 = \omega(\beta t\eta_2)/\beta t$ and applying the change of variables $t\eta_2 = x$ to the inner integral and observing that

$$\int_0^1 \Big(\int_{1/\beta}^{\infty} (1/t^2)(\exp(-zx - x/t))\,dx\Big)\,d\mu(t)$$

$$= \exp(-z/\beta)\int_0^1 (\exp(-1/\beta t))/t(1+zt)\,d\mu(t)$$

we obtain the first part of the theorem. For the proof of the second part of the theorem we have only to observe that if $\beta > 0$ and $\operatorname{Re}(z) \geqslant -1+\delta$, where $0 < \delta < 1$, then, since $\eta_1 \leqslant \eta_2$, we have that

$$|\exp(-zt\eta_1) - \exp(-zt\eta_2)|\exp(-\eta_2) \leqslant \exp(-\delta\eta_2),$$

and so,

$$|f(z) - A_f(z;\beta)| \leqslant \int_0^1 (1/t)\Big(\int_{1/\beta t}^{\infty} \exp(-\delta\eta_2)\, d\eta_2\Big)\, d\,|\mu|(t)$$

$$= (1/\delta) \int_0^1 (1/t) \exp(-\delta/\beta t)\, d\,|\mu|(t),$$

and the proof is finished.

Remarks 1. The function $\exp(-z/\beta) \int_0^1 (\exp(-1/\beta t)/t(1 + zt))\, d\mu(t)$ $\mathrm{Re}(z) > -1$, can be extended analytically into the complex plane minus the negative real numbers $\leqslant -1$. Since f and A_f are both analytic in that domain it follows that the function $\int_0^1 (1/t^2)(\int_{1/\beta}^{\infty} \exp((-z\omega(\beta x)/\beta - x/t)\,(dx)\, d\mu(t)$ can be extended analytically to the same domain.

2. If $0 < \beta < \delta$, then $\int_0^1 (1/t) \exp(-\delta/\beta t)\, d\,|\mu|(t) \leqslant \exp(-\delta/\beta) \int_0^1 d\,|\mu|$. Furthermore, by observing that for each $a > 0$ and for each $k = 1, 2,\ldots$ the function $u^k \exp(-au)$ attains its absolute maximum at $u = k/a$ in $u \geqslant 0$, we obtain that for all z satisfying $\mathrm{Re}(z) \geqslant -1 + \delta$, $0 < \delta < 1$, $\beta > 0$ and for all $k = 1, 2,\ldots$ $|f(z) - A_f(z, \beta)| \leqslant ((1/\delta^{k+1})\, k^k e^{-k} \int_0^1 t^{k-1}\, d\,|\mu|)\, \beta^k$.

4. Examples

1. In [2] Viggo Brun posed the problem of determining the sum s of the infinite series $\sum_{n=0}^{\infty} n^n/(n+2)^{n+2}$, where we set $0^0 = 1$. The first correct answer was presented by O. Kolberg in [7]. Kolberg arrived at his answer by observing that the infinite series is related to the Abel series generated by a simple rational function. For the sake of completeness we shall present a solution based on Theorem 3.11.

Consider the rational function $f(z) = z/2(z+2)$. It is easy to see that $f \in \mathcal{Ol}$. Indeed, we have $z/2(z+2) = \int_0^1 z/(1+zt)\, d\mu(t)$, where μ is the discrete measure concentrated at $t = \frac{1}{2}$ with total measure $\frac{1}{4}$. It is not difficult to see that the Abel series generated by f has the following form

$$A_f(z;\beta) = \sum_{n=1}^{\infty} (-1)^{n-1} z(z - n\beta)^{n-1}/(2 + n\beta)^{n+1}. \tag{4.1}$$

Observe now that

$$s = 2 \sum_{n=0}^{\infty} n^n/(n+2)^{n+2} = 2A_f(2, 2).$$

Hence, $s = 2(A_f(2, 2) - f(2)) + \frac{1}{2}$. According to Theorem 3.11

$$A_f(2, 2) - f(2) = -\tfrac{1}{2}(\tfrac{1}{2}) \int_1^\infty (\exp(-\omega(\xi) - \xi) - \exp(-2\xi))\, d\xi$$

$$= e^{-2}/4 - (\tfrac{1}{2}) \int_1^\infty \exp(-\omega(\xi) - \xi)\, d\xi,$$

and so, we obtain the following formula

$$s = \sum_{n=0}^{\infty} n^n/(n+2)^{n+2} = \tfrac{1}{2} + e^{-2}/2 - \int_1^\infty \exp(-\omega(\xi) - \xi)\, d\xi. \quad (4.2)$$

In order to obtain Kolberg's formula we set, following Kolberg, $x = \exp(-\xi)$ and $y = \exp(-\omega(\xi))$, then x and y satisfy the relation $x^x = y^y$. Hence, if we set $x/y = t$, then x and y can be expressed in terms of t by the formulas: $x = t^{1/(1-t)}$ and $y = t^{t/(1-t)}$. Observe now that, if x increases from 0 to $1/e$, then t increases from 0 to 1; and if t tends to 0, then $x(t)$ tends to 0 and $y(t)$ tends to 1; and, if t tends to 1, then $x(t)$ and $y(t)$ both tend to $1/e$. Furthermore, if t increases from 0 to 1, then y decreases from 1 to $1/e$.

We conclude that $\int_1^\infty \exp(-\omega(\xi) - \xi)\, d\xi = \int_0^{1/e} y\, dx = \int_0^1 (x(t)/t) dx(t) = (\frac{1}{2}) \int_0^1 (1/t)\, dx^2(t) = e^{-2}/2 + (\frac{1}{2}) \int_0^1 x^2(t)/t^2\, dt = e^{-2}/2 + (\frac{1}{2}) \int_0^1 t^{2t/(1-t)}\, dt$, and so, finally we obtain Kolberg's result namely:

$$s = \sum_{n=0}^{\infty} n^n/(n+2)^{n+2} = \Big(1 - \int_0^1 t^{2t/(1-t)}\, dt\Big)/2 = \tfrac{1}{2} - \int_0^\infty t^t/(t+2)^{(t+2)}\, dt. \quad (4.3)$$

Formula (4.3) can also be obtained from (4.2) by remarking that if we write $\omega(\xi) = \xi\gamma(\xi)$, $\xi \geqslant 1$, then it follows immediately from $\omega(x)\exp(-\omega(x)) = x\exp(-x)$, $x \geqslant 1$ that γ is the inverse function of the function $\xi = (\log \eta)/(\eta - 1)$, $0 < \eta \leqslant 1$, and so, $\gamma((\log \eta)/(\eta - 1)) = \eta$, $0 < \eta \leqslant 1$. Observing that $\int_1^\infty \exp(-\omega(\xi) - \xi)\, d\xi = \int_1^\infty \exp(\xi - \xi\gamma(\xi)) \exp(-2\xi)\, d\xi$ and substituting $\xi = \varphi(t) = (\log t)/(t - 1)$ we obtain that

$$\int_1^\infty \exp(-\omega(\xi) - \xi)\, d\xi$$

$$= -\int_0^1 (1/t) \exp(-2\varphi(t))\, d\varphi(t) = \tfrac{1}{2} \int_0^1 (1/t)\, d(\exp(-2\varphi(t))$$

$$= e^{-2}/2 + \int_0^1 t^{-2} \exp(-2\varphi(t))\, dt = e^{-2}/2 + \int_0^1 t^{2t/(t-1)}\, dt,$$

and again we have arrived at (4.3).

2. If $\mu(t) = t$ is the Lebesgue measure, then $f(z) = \log(1 + z) \in \mathcal{A}$. In this case, $f^{(n)}(z) = (-1)^{n-1}(n-1)!\,(1+z)^{-n}$, $n = 1, 2,\ldots$, and so, if $\beta > 0$, then

$$A_f(z;\beta) = \sum_{n=1}^{\infty} (-1)^{n-1} z(z - n\beta)^{n-1}/n(1 + n\beta)^n. \tag{4.4}$$

Hence, by Theorem 3.11, we have for all $\beta > 0$ and for all z satisfying $\mathrm{Re}(z) > -1$

$$\sum_{n=1}^{\infty} (-1)^{n-1} z(z - n\beta)^{n-1}/n(1 + n\beta)^n$$
$$= \log(1 + z) + \exp(-(z + 1)/\beta) \int_0^{\infty} \exp(-t/\beta)/(t + z + 1)\, dt$$
$$- \int_0^1 (1/t^2)\left(\int_{1/\beta}^{\infty} \exp(-z\omega(\beta x)/\beta - x/t)\, dx\right) dt. \tag{4.5}$$

From

$$\int_0^1 (1/t^2)\left(\int_{1/\beta}^{\infty} \exp(-z\omega(\beta x)/\beta - x/t)\, dx\right) dt = \int_1^{\infty} (\exp(-z\omega(x)/\beta - x/\beta)/x)\, dx$$

we obtain the formula

$$\sum_{n=1}^{\infty} (-1)^{n-1} z(z - n\beta)^{n-1}/n(1 + n\beta)^n$$
$$= \log(1 + z) + \exp(-(z + 1)/\beta) \int_0^{\infty} \exp(-t/\beta)/(1 + z + t)\, dt$$
$$- \int_1^{\infty} (\exp(-z\omega(x)/\beta - x/\beta)/x)\, dx. \tag{4.6}$$

From (3.13) it follows, in particular, that for all z satisfying $\mathrm{Re}(z) \geqslant 1 + \delta$, where $\delta > 0$, we have

$$\left| \sum_{n=1}^{\infty} (-1)^{n-1} z(z - n\beta)^{n-1}/n(1 + n\beta)^n - \log(1 + z)\right| \leqslant (\beta/\delta^2) \exp(-\delta/\beta). \tag{4.7}$$

If we put $z = 2$ and $\beta = 1$ in (4.6), then we obtain the following corrected version of a formula due to Abel [1, Second edition, p. 74]

$$\sum_{n=1}^{\infty} (n - 2)^{n-1}/n(n + 1)^n = \log \sqrt{3} + (\tfrac{1}{2}) \int_0^{\infty} (\exp(-(3 + t))/(3 + t))\, dt$$
$$- (\tfrac{1}{2}) \int_1^{\infty} (1/x) \exp(-2\omega(x) - x)\, dx. \tag{4.8}$$

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